Multilevel Monte Carlo Methods for UQ

Appendix: Proofs

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Proof of Lemma 4.1:

$$\mathbb{E}\left[\left(\hat{Q}_h - \mathbb{E}(Q)\right)^2\right] = \mathbb{E}\left[\left(\underbrace{\hat{Q}_h - \mathbb{E}(Q_h)} + \underbrace{\mathbb{E}(Q_h) - \mathbb{E}(Q)}\right)^2\right]$$

$$= \mathbb{E}\left[\left(\hat{Q}_h - \mathbb{E}(\hat{Q}_h)\right)^2\right] + \cancel{\mathbb{E}}\left[\left(\mathbb{E}(Q_h) - \mathbb{E}(Q)\right)^2\right]$$

$$+ 2\,\mathbb{E}\left[\left(\hat{Q}_h - \mathbb{E}(\hat{Q}_h)\right)\left(\mathbb{E}(Q_h) - \mathbb{E}(Q)\right)\right]$$

$$= \mathrm{Var}(\hat{Q}_h) + \left(\mathbb{E}(Q_h - Q)\right)^2 + 0$$

$$= \frac{\mathrm{Var}(Q_h)}{N} + \left(\mathbb{E}(Q_h - Q)\right)^2$$

Proof of Thm 5.2. Lagrange-multipliers. Let

$$\mathcal{L}(N_0,\dots,N_L,\lambda) = \sum_{\ell=0}^{L} N_\ell C_\ell + \lambda\left(\sum_{\ell=0}^{L}\frac{\overset{=\mathrm{Var}(Y_\ell)}{V_\ell}}{N_\ell} - \frac{\varepsilon^2}{2}\right)$$

1st order optim.:

$$0 = \frac{\partial\mathcal{L}}{\partial\lambda} = \sum_{\ell=0}^{L}\frac{V_\ell}{N_\ell} - \frac{\varepsilon^2}{2} \qquad (a)$$

$$0 = \frac{\partial\mathcal{L}}{\partial N_\ell} = C_\ell - \lambda\frac{V_\ell}{N_\ell^2} \qquad (b)$$

$$\overset{(b)}{\Longrightarrow} \quad N_\ell = \sqrt{\lambda}\sqrt{\frac{V_\ell}{C_\ell}}, \quad \ell = 0,\dots,L$$

To find λ $\longrightarrow$ substitute into (a).

To ensure $\left(\mathbb{E}\left[\hat{Q}_L^{ML} - Q\right]\right)^2 < \frac{\varepsilon^2}{2}$ it suffices to choose L s.t.

$$m^{-\alpha}\frac{\varepsilon}{\sqrt{2}} < c_1 m^{-\alpha L} \le \frac{\varepsilon}{\sqrt{2}} \qquad (c)$$

$$\left(\text{thus } L = \left\lceil \alpha^{-1}\log_m\left(\sqrt{2}c_1\varepsilon^{-1}\right)\right\rceil\right)$$

To bound variance of estimator by $\frac{\varepsilon^2}{2}$, first use (left ineq. in (c)) $\Longrightarrow$ (geometric series)

$$\sum_{\ell=0}^{L} m^{\gamma\ell} < \frac{m^{\gamma L}}{1-m^{-\gamma}} < \underbrace{\frac{m^{\gamma}\left(\sqrt{2}c_1\right)^{\gamma/\alpha}}{1-m^{-\gamma}}}_{=:c_4\ (=\text{const})}\varepsilon^{-\gamma/\alpha} \qquad (d)$$

Now need to distinguish three cases:

a) $\beta = \gamma$: Set $N_\ell = \left\lceil 2\varepsilon^{-2}(L+1)\,c_2\,m^{-\beta\ell}\right\rceil$

$$\Rightarrow \mathrm{Var}\left[\hat{Q}_L^{ML}\right] = \sum_{\ell=0}^{L}\frac{\mathrm{Var}(Y_\ell)}{N_\ell} \le \sum_{\ell=0}^{L} c_2 N_\ell^{-1} m^{-\beta\ell} \le \frac{\varepsilon^2}{2}$$

$N_\ell = \lceil x_\ell \rceil \le x_\ell + 1$

$$\Rightarrow \mathrm{Cost}\left(\hat{Q}_L^{ML}\right) \le c_3\sum_{\ell=0}^{L} N_\ell m^{\gamma\ell} \le c_3\left(2\varepsilon^{-2}(L+1)^2 c_2 + \sum_{\ell=0}^{L} m^{\gamma\ell}\right) \qquad (e)$$

$$\varepsilon < e^{-1} < 1 \quad\Rightarrow\quad 1 < \log(\varepsilon^{-1})$$

$$\alpha \geq \tfrac{1}{2}\gamma \quad \Rightarrow \quad \varepsilon^{-\gamma/\alpha} \leq \varepsilon^{-2} \leq \varepsilon^{-2}\left(\log(\varepsilon^{-1})\right)^2$$

$$\overset{(c\text{-}e)}{\Rightarrow} \qquad \mathrm{Cost}\left(\hat{Q}_L^{ML}\right) = \Theta\left(\varepsilon^{-2}\left(\log \varepsilon\right)^2\right)$$

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b) $\beta < \gamma$: Set $N_\ell = \left\lceil \sqrt{\lambda}\,\sqrt{V_\ell/C_\ell}\right\rceil = \left\lceil 2\varepsilon^{-2} c_2\, m^{(\gamma-\beta)L/2}\, \dfrac{m^{-(\beta+\gamma)\ell/2}}{1-m^{-(\gamma-\beta)/2}}\right\rceil$

$$\Rightarrow \sum_{\ell=0}^{L} \frac{\overset{\leq c_2 m^{-\beta\ell}}{\mathrm{Var}[Y_\ell]}}{N_\ell} < \frac{\varepsilon^2}{2} \underbrace{m^{-\frac{(\gamma-\beta)L}{2}}\left(1-m^{-\frac{(\gamma-\beta)}{2}}\right)\sum_{\ell=0}^{L} m^{(\gamma-\beta)\ell/2}}_{=1 \ (\text{geometric series})} = \frac{\varepsilon^2}{2}$$

and

$$\mathrm{Cost}\left(\hat{Q}_L^{ML}\right) \leq c_3\left(2\varepsilon^{-2}c_2 \frac{m^{(\gamma-\beta)L/2}}{1-m^{-(\gamma-\beta)/2}} \sum_{\ell=0}^{L} m^{(\gamma-\beta)\ell/2} + \sum_{\ell=0}^{L} m^{\gamma\ell}\right)$$

$$\leq c_3\left(2\varepsilon^{-2}c_2 \frac{m^{(\gamma-\beta)L}}{\left(1-m^{-(\gamma-\beta)/2}\right)^2} + \sum_{\ell=0}^{L} m^{\gamma\ell}\right) \qquad (f)$$

Using the first ineq. in (c) $\Rightarrow$

$$m^{(\gamma-\beta)L} \leq \left(\sqrt{2}\,c_1\right)^{(\gamma-\beta)/\alpha} m^{\gamma-\beta}\, \varepsilon^{-\frac{(\gamma-\beta)}{\alpha}}.$$

Also $\left.\begin{array}{l}\varepsilon < e^{-1} < 1 \\ \alpha \geq \tfrac{1}{2}\beta\end{array}\right\} \Rightarrow \varepsilon^{-\gamma/\alpha} \leq \varepsilon^{-2-\frac{\gamma-\beta}{\alpha}}$

Using both those bounds as well as (d) in (f) we finally get

$$\mathrm{cost}\left(\hat{Q}_L^{ML}\right) \leq \frac{2c_2c_3}{\left(1-m^{-(\gamma-\beta)/2}\right)^2}\left(\sqrt{2}\,c_1\right)^{(\gamma-\beta)/\alpha} m^{\gamma-\beta}\, \varepsilon^{-2-\frac{\gamma-\beta}{\alpha}} + c_3c_4\,\varepsilon^{-\frac{\gamma}{\alpha}}$$

$$= \Theta\left(\varepsilon^{-2-\frac{\gamma-\beta}{\alpha}}\right)$$

$\beta > \gamma$: similar.

For a detailed proof see [Cliffe, Giles, RS, Teckentrup, 2011, Appendix]. $\square$